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SHEAR INSTABILITY IN STRATIFIED VISCOELASTIC AND GAS-LIQUID MEDIA*

I.SH. AKHATOV

A number of problems on the development of surface and internal waves in fluids with complex rheophysical properties is solved in a linear formulation. The influence of the viscoelastic properties of non-Newtonian fluids and the dissipation properties and local deformation inertia of fluids with gas bubbles on the stability of such waves is investigated.

The question of the stability of internal waves in a two-layered incompressible viscous fluid with a tangential velocity discontinuity has been studied in detail /1, 2/. It is known that a Kelvin-Helmholtz instability occurs in the short-wave domain while there is no instability in the long-wave domain because of the stabilizing action of stratification. However, waves with negative energies exist in this domain that become unstable in the presence of viscous dissipation. In this connection, investigation of the stability of waves with negative energies in two-layered non-Newtonian fluids characterized by more complex rheological behaviour is of interest.

1. Waves on the surface of a viscoelastic fluids. A viscoelastic incompressible medium of infinite depth $(\infty < y < 0)$ is considered in an x, y Cartesian system of coordinates. The free fall acceleration $g = \text{const}$ is directed along the negative y semi-axis.

The generalized rheological law

$$\mathbf{R}\tau_{ij} = \mathbf{Q}\varepsilon_{ij}^{\cdot} \tag{1.1}$$

relating the components of the deviator part of the stress tensor τ_{ij} to the components of the strain rate tensor $\varepsilon_{ij}^{\cdot}$ is used. Here R and Q are differential operators based on the Oldroyd derivative with respect to time /3/ which can be written in the linear approximation in the form

$$\mathbf{R} = 1 + \theta\partial/\partial t, \ \mathbf{Q} = 2\mu\,(1 + \lambda\partial/\partial t) \tag{1.2}$$

where μ is the coefficient of dynamic viscosity and θ and λ are the relaxation times. It can be shown that the system of linearized equations of the two-dimensional non-stationary motion of a viscoelastic fluid has the following form when relationships (1.1) and (1.2) are used:

$$\mathbf{R}\,(\rho\partial u/\partial t + \partial p/\partial x) = \frac{1}{2}\mathbf{Q}\Delta u, \quad \mathbf{R}\,(\rho\partial v/\partial t + \partial p/\partial y) = \tag{1.3}$$
$$\frac{1}{2}\mathbf{Q}\Delta v - \rho g, \quad \partial u/\partial x + \partial v/\partial y = 0$$

The equation of the fluid surface $y = \eta\,(x, t)$ is related to the velocity field by the standard relationship

$$\partial\eta/\partial t - v = 0, \ y = 0 \tag{1.4}$$

The dynamic boundary conditions that the tangential and normal stress components equal zero on the fluid surface yield, in the linear approximation,

$$\mu\,(\partial u/\partial y + \partial v/\partial x) = 0, \ \mathbf{R}p - \mathbf{Q}\partial v/\partial y = 0, \ y = 0 \tag{1.5}$$

The solution of Problem (1.2)-(1.5) can be represented in the form /4/

$$\begin{gathered} u = u_0 - \partial\psi/\partial y, \ v = v_0 + \partial\psi/\partial x, \ p = p_0 \\ u_0 = ikBE\,(t, x, y), \ v_0 = kBE\,(t, x, y) \\ p_0 = i\omega\rho BE\,(t, x, y) - \rho gy; \ E\,(t, x, y) = \\ \exp\,(-i\omega t + ikx + ky) \end{gathered} \tag{1.6}$$

where u_0, v_0, p_0 is the solution of this problem for an ideal incompressible fluid and $(\mu = 0, \lambda = 0, \theta = 0)$, ψ is a function of x, y, t. Substitution of these expressions into the first

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two equations of (1.3) when taking account of the equalities $\Delta u_0 = 0, \Delta v_0 = 0$ results in the equation

$$2\rho \mathbf{R} \partial\psi/\partial t = \mathbf{Q}\Delta\psi$$

whose solution has the form

$$\psi = C \exp(-i\omega t + ikx + ly)$$
$$l^2 = k^2 - \frac{i\omega r(\omega)}{\nu q(\omega)}, \quad r(\omega) = 1 - i\omega\theta, \quad q(\omega) = 1 - i\omega\lambda, \quad \nu = \frac{\mu}{\rho}$$

We consequently obtain

$$u = (ikB \exp(ky) - lC \exp(ly)) \exp(-i\omega t + ikx) \tag{1.7}$$
$$v = (kB \exp(ky) + ikC \exp(ly)) \exp(-i\omega t + ikx), \; p = p_0$$

Further utilization of the boundary Conditions (1.4) and (1.5) results in the dispersion relation

$$\omega^2 - \omega_0{}^2 + 4i\nu k^2\omega q(\omega)/r(\omega) + 4\nu^2 k^4 (l/k - 1)(q(\omega)/r(\omega))^2 = 0 \tag{1.8}$$

where $\omega_0{}^2(k) = gk$ is the dispersion relation for waves on the surface of an ideal incompressible fluid. In the case of small viscosities $(\nu k^2/\omega \ll 1)$ it can be assumed that the motion differs little from the potential flow of an ideal fluid so that $\omega = \omega_0 + i\delta$, $\delta \ll \omega_0$.

The solution of (1.8) in the linear approximation in δ has the form

$$\delta = -2\nu k^2 [1 + gk\lambda\theta + i\sqrt{gk}(\theta - \lambda)]/(1 + gk\theta^2)$$

It is seen that the presence of viscoelastic properties in the fluid results in a change in the dispersion and dissipation characteristics. Thus, we have for a phase velocity c and a damping decrement $-\operatorname{Re}\delta$

$$c = \sqrt{g/k}\,[1 + 2\nu k^2(\theta - \lambda)/(1 + gk\theta^2)]$$
$$-\operatorname{Re}\delta = 2\nu k^2(1 + gk\lambda\theta)/(1 + gk\theta^2)$$

The equality $\omega_0{}^2 = gk + \sigma k^3/\rho$ must be used to take account of the influence of the surface tension σ in (1.8).

2. *Internal waves in a laminar viscoelastic fluid.* Two incompressible fluids of different densities ae examined, where the lower one $(y < 0)$ is viscoelastic, has a high density and is fixed, while the upper one $(y > 0)$ is ideal and moves with velocity U. The parameters referring to the upper and lower fluids are marked with the subscripts 1 and 2, respectively.

The kinematic and dynamic conditions on the interfacial boundary of the liquids result in the equations

$$\partial\eta/\partial t + U\partial\eta/\partial x - v_1 = 0, \; \partial\eta/dt - v_2 = 0$$
$$\mu(\partial u_2/\partial y + \partial v_2/\partial x) = 0, \; \mathbf{R}(p_2 - p_1) = \mathbf{Q}\partial v_2/\partial y \tag{2.1}$$

Relations (1.7) obtained in Sect.1 are used as solutions of the equations of the dynamics of viscoelastic fluids (1.3). The solution for the upper (ideal) fluid can be represented in the form

$$u_1 = U + ikAE(t, x, -y), \; v_1 = -kAE(t, x, -y)$$
$$p_1 = -i\rho_1 A(\omega - Uk)E(t, x, -y) - \rho_1 gy \tag{2.2}$$

Substitution of relations (1.7) and (2.2) into (2.1) yields the following dispersion relation:

$$s(\omega - Uk)^2 + \omega^2 - (1 - s)gk + 4i\nu k^2\omega q(\omega)/r(\omega) +$$
$$4\nu^2 k^4 (l/k - 1)(q(\omega)/r(\omega))^2 = 0, \; s = \rho_1/\rho_2 \tag{2.3}$$

For $\nu k^2/\omega \ll 1$ (2.3) can be reduced to a dispersion relation for internal waves in a laminar waves in a laminar ideal fluid $Z_0(\omega, k) = 0$ with a correction due to the rheological constants ν, λ, θ

$$Z_0(\omega, k) = -iF(k, \theta, \lambda, \omega, k) \tag{2.4}$$
$$Z_0 = s(\omega - Uk)^2 + \omega^2 - (1 - s)gk$$

$$F = 4\nu k^2\omega[1 + \omega^2\lambda\theta + i\omega(\theta - \lambda)]/(1 + \omega^2\theta^2)$$

In a laminar ideal fluid described by the dispersion equation $Z_0(\omega, k) = 0$, the domain of Kelvin-Helmholtz instability $(k > k_2 = (1 - s^2)g/(sU^2))$ bounds the negative-energy wave domain located between the critical points $k_1 = (1 - s)g/(sU^2)(\omega = 0)$ and $k_2(\partial\omega/\partial k = \infty)$ /1, 2/ (Fig.1). Indeed, the average wave energy density is given in the linear approximation by the relationship

$$E = \omega a^2 \partial Z_0/\partial w = 2\omega a^2(1 + s)(\omega - Us(1 + s)^{-1}k)$$

from which it follows that the branch A_1A_2 of the dispersion curve lying below the line OE $(\omega = Us(1 + s)^{-1}k)$, corresponds to waves with negative energy. It is well-known that if the lower medium is a Newtonian viscous fluid, then the Kelvin-Helmholtz instability is weakened while an instability /1, 2/ (dissipative instability) occurs in the whole domain of negative-energy waves.

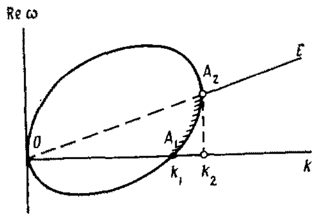

Fig.1

A small correction $i\delta$ to the frequency $(\omega = \omega_0 + i\delta)$: is found from (2.4)

$$\delta = -\frac{2\nu k^2\omega}{1+s}\left(\omega - \frac{sU}{1+s}k\right)^{-1}(1 + \omega^2\theta^2)^{-1}[1 + \omega^2\lambda\theta + i\omega(\theta - \lambda)]$$

It is hence seen that if $\theta = 0$, the growth increment $\operatorname{Re}\delta$ of the negative-energy waves is the same as in the case of a Newtonian viscous fluid. If $\lambda < \theta \neq 0$, then the growth increment is lower than for a Newtonian fluid while if $\lambda > \theta \neq 0$, it is above. Therefore, the presence of visco-elastic properties in a fluid can result in both magnification and attenuation of the dissipative instability, depending on the magnitudes of the relaxation times.

3. ***Waves on a fluid surface with gas bubbles.*** Gas-liquid media are fluids with a non-holonomic equation of state $p = f(\rho, \dot\rho, \ddot\rho)$ /5, 6/ (the dot denotes the Lagrange time derivative). In the linear approximation the pressure deviation p' and the density deviation ρ' from their initial equilibrium values p_0 and ρ_0 are connected by the relationship

$$p' = c_0^2\rho' + \alpha\dot\rho' + \beta\ddot\rho' \tag{3.1}$$

$$c_0^2 = \frac{\gamma p_0}{\rho^\circ\varphi_0}, \quad \alpha = \frac{4\mu}{3\varphi_0\rho^\circ}, \quad \beta = \frac{a^2}{3\varphi_0}$$

Here γ is the polytropic gas index in the bubbles, φ_0 and a are the volume concentration and the radius of the bubbles, and μ is the effective viscosity. Fine-scale phyiscochemical processes occurring in gas-liquid media and their influence on the dissipative properties (the parameter α) and the dispersion (the parameter β) of the mixture are analysed in detail in /6/.

In the initial unperturbed state the liquid density distribution with depth is determined by the formula

$$\rho_0(y) = \rho_0(0)\exp(-gy/c_0^2)$$

consequently, the fluid can be considered homogeneous ($\rho_0 = \text{const}$) with good accuracy for wavelengths L satisfying the condition $gL/c_0^2 \ll 1$.

After linearization the momenta continuity equations have the form

$$\partial\rho'/\partial t + \rho_0 \operatorname{div}\mathbf{v}' = 0, \quad \rho_0\partial\mathbf{v}'/\partial t + \nabla p' - \rho'\mathbf{g} = 0 \tag{3.2}$$

The last component in the second equation in (3.2) can be neglected by taking into account that the density perturbation is negligibly small compared with the pressure gradient for $gL/c_0^2 \ll 1$. Then, taking (3.1) into account we obtain an equation from (3.2) for the potential flow $(\mathbf{v}' = \nabla\varphi)$

$$\frac{\partial^2\varphi}{\partial t^2} - \left(c_0^2 + \alpha\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right)\Delta\varphi = 0 \tag{3.3}$$

which the travelling wave

$$\varphi = A\exp(-i\omega t + ikx + iq(\omega)y) \tag{3.4}$$

$$q^2(\omega) = -k^2 + \omega^2(c_0^2 - i\alpha\omega - \beta\omega^2)^{-1}$$

satisfies.

Utilization of standard boundary conditions for an ideal fluid results in the dispersion relation

$$\omega^4 + g^2 q^2(\omega) = 0 \tag{3.5}$$

For $\alpha = 0,\ \beta = 0$, Eq.(3.5) reduces to the equation

$$Z_0(\omega_0, k) = \omega_0^4 - g^2(k^2 - \omega_0^2 c_0^{-2}) = 0$$

whose solution possesses the following asymptotic properties:

$$\omega_0 = c_0 k, \quad k \to 0; \quad \omega_0 = \sqrt{gk}, \quad k \to \infty$$

For small α and β, dispersion relation (3.5) can be represented in the form

$$Z_0(\omega, k) = -i\alpha g^2 \omega^3 c_0^{-4} - \beta g^2 \omega^4 c_0^{-4}$$

Hence, for a small correction $i\delta$ to the frequency the expression

$$\delta = \frac{-\alpha g^2 \omega_0^2 + i\beta g^2 \omega_0^3}{2(2c_0^4 \omega_0^2 + g^2 c_0^2)}$$

is obtained.

It is seen that waves on the surface of a fluid with gas bubbles damp out with a damping decrement that tends to zero according to the law $-\operatorname{Re}\delta = \alpha k^2/2$, as $k \to 0$ and tends to a finite value $-\operatorname{Re}\delta = \alpha g^2/(4c_0^4)$ as $k \to \infty$

4. Waves on the interfacial boundary of bubbly and ideal fluids.
An ideal incompressible fluid of infinite depth $(y < 0)$ is considered above which $(y > 0)$ is a fluid with gas bubbles moving horizontally at the velocity U. The parameters referring to the upper and lower fluids are marked with the subscripts 1 and 2, respectively. The equation for the perturbation potential in the bubbly fluid φ_1 is obtained from (3.3) by the operational replacement $\partial/\partial t \to \partial/\partial t + U\partial/\partial x$ and its solution is obtained from (3.4) by the replacement $\omega \to (\omega - Uk)$ in the expression for $q(\omega)$. Utilization of the boundary conditions

$$(\partial/\partial t + U\partial/\partial x)\eta - \partial\varphi_1/\partial y = 0, \ \partial\eta/\partial t - \partial\varphi_2/\partial y = 0$$
$$\rho_1 g\eta + \rho_1(\partial/\partial t + U\partial/\partial x)\varphi_1 = \rho_2 g\eta + \rho_2 \partial\varphi_2/\partial t$$

and the solutions φ_2 of the equations of ideal incompressible fluid motion results in a dispersion relation

$$\omega^2 + ikq^{-1}(\omega)\, s(\omega - Uk)^2 - (1 - s)gk = 0$$

that can be reduced to the dispersion relation for internal waves in a laminar ideal fluid $Z_0(\omega, k) = 0$ in the linear approximation (see (2.4)) with a correction due to the compressibility (c_0^{-2}), and the dissipative property (α) and dispersion (β) of the gas-liquid medium

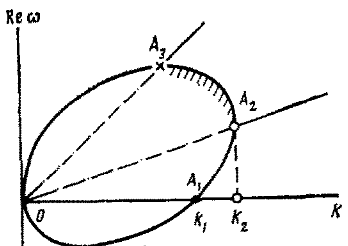

Fig.2

$$Z_0 = -\frac{s(\omega - Uk)^4}{2k^2 c_0^2}\left[1 + \frac{\beta}{c_0^2}(\omega - Uk)^2\right] - i\,\frac{\alpha s(\omega - Uk)^5}{2k^2 c_0^4} \tag{4.1}$$

An expression is obtained from (4.1) for the wave growth increment

$$\operatorname{Re}\delta = -\frac{\alpha s(\omega - Uk)^5}{4(1+s)k^2 c_0^4\left(\omega - \dfrac{sU}{1+s}k\right)}$$

from which it follows that in a domain bounded by the line $\omega = Uk$ from above and the line $\omega = s(1 + s)^{-1}Uk$ from below, a dissipative wave instability (shaded in Fig.2) should occur irrespective of the magnitude of the parameter α.

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A VARIATIONAL PRINCIPLE FOR NON-LINEAR CONCENTRATED WAVES*

S.A. VAKULENKO

Asymptotic solutions describing domain walls and wave beams in a non-linear continuous medium are considered. The shape of the walls or beams can be derived from a simple variational principle - a generalization of Fermat's principle in linear geometrical optics to the non-linear situation.

1. Statement of the problem. We consider asymptotic solutions for certain classes of non-linear equations. The following equations will be studied:

$$\Delta u+\omega^2 V_u'(u,\mathbf{x})=0,\quad u(\mathbf{x}):R^m\rightarrow R,\quad \mathbf{x}\in R^m \tag{1.1}$$

and also

$$\Delta u+\omega^2\Phi_{|u|^2}'(|u|^2,\mathbf{x})u=0,\quad u(\mathbf{x})\in C,\quad \mathbf{x}\in R^k \tag{1.2}$$

where $\omega\gg 1$, $m\geqslant 1$, $k=2,3$.

Eq.(1.1) has applications (when $m=2$) in two-dimensional problems of elasticity theory for liquid crystals (these applications will be considered in Sect.4). Eq.(1.2) is used to describe the propagation of radiation in a non-linear medium /1, 2/. In that case u is the complex amplitude of the field and Φ' is the non-linear refractive index.

Special asymptotic solutions (as $\omega\rightarrow\infty$) of Eqs.(1.1), (1.2) were considered in /3/. In this paper, for brevity, we shall use the term "concentrated solution" (for a rigorous definition see /3/).

The following is an example of a concentrated solution. Put $m=1$, $\alpha>0$, $V_u'=\alpha^2(x)\sin u$ in (1.1). Then the equation has an asymptotic solution

$$u=4\operatorname{arctg}(\exp(\omega\alpha(x_0)(x-x_0))+O(\omega^{-1})=u_0(x)+O(\omega^{-1}) \tag{1.3}$$

The function $u_0(x)$ varies essentially in a narrow region, of size $O(\omega^{-1})$, near the point x_0, but when $|x-x_0|\gg\omega^{-1}$ the function $u_0(x)$ differs by an exponentially small amount from 0 or 2π. The solution is concentrated near x_0. When $m>1$ such solutons of Eq.(1.1) are concentrated near hypersurfaces S in R^m; similar solutions of Eq.(1.2) concentrate near curves l. Solutions of Eq.(1.1) are interpreted as domain walls, and those of

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